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Squeezing and Dynamical Symmetries

José M. Cerveró[1]

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A systematic formalism for dealing with nonrelativistic time-dependent quantum Hamiltonians is presented. The starting point is the Lewis and Riesenfeld invariant isospectral operator $I(x, t)$. We discuss three examples: the generalized harmonic oscillator, the conformal oscillator, and the infinite square well with a moving boundary. We obtain the same results for the generalized harmonic oscillator as other approaches. In the case of the square well with a moving boundary, an effective interaction appears which seems to be due to the time dependence of the boundary. Consistency with the principle of minimal coupling and gauge invariance is obtained. Some interesting physical applications are suggested.

1. INTRODUCTION

Since the discovery of electromagnetic traps for charged and neutral particles by W. Paul enormous effort has been dedicated to the construction of a consistent formalism for time-dependent quantum systems. The reason is that, as shown by various authors, the Paul trap can be quite accurately described by a time-dependent harmonic oscillator. More recently, Sutherland proposed to describe trapped Bose–Einstein condensates by means of a generalized time-dependent Calogero–Sutherland model which contains harmonic terms mixed with centrifugal barriers in a pairwise interaction. This increasingly important experimental area calls for a detailed exact quantum time-dependent formalism describing accurately the physical properties of matter trapped at very low temperatures and strong varying magnetic fields. The main purpose of this paper is to describe and discuss such a formalism.

Given a Hamiltonian operator $H(x, t)$, which we shall be taking for simplicity in the coordinate representation, Lewis and Riesenfeld[(1)] showed that, it is possible to build an invariant operator satisfying

[1] Area de Física Teórica, Facultad de Ciencias, Universidad de Salamanca, 37008 Salamanca, Spain.

$$\frac{dI(x,t)}{dt} = \frac{\partial I(x,t)}{\partial t} + \frac{1}{i\hbar}[I(x,t), H(x,t)] \tag{1}$$

$$I(x,t) = I^{+}(x,t) \tag{2}$$

such that the operator $I(x, t)$ is isospectral, that is, its eigenvalues ϵ_n are constants. The wave functions $\Phi_n(x, t)$ of the invariant operator $I(x, t)$

$$I(x,t)\Phi_n(x,t) = \epsilon_n \Phi_n(x,t) \tag{3}$$

and those of the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi_n(x,t) = H(x,t)\Psi_n(x,t) \tag{4}$$

are related through

$$\Psi_n(x,t) = e^{i\alpha_n(t)}\Phi_n(x,t) \tag{5}$$

The so-called *Lewis phases* $\alpha_n(t)$ can be found[3] as

$$\begin{aligned}\alpha_n(t) = {} & \int_R dx \left[-\frac{1}{\hbar} \int_0^t \Phi_n^*(x,s) H(x,s) \Phi_n(x,s)\, ds \right] \\ & + i \int_R dx \left[\int_0^t \Phi_n^*(x,s) \frac{\partial}{\partial s} \Phi_n(x,s)\, ds \right] \end{aligned} \tag{6}$$

2. THE GENERALIZED HARMONIC OSCILLATOR

In this section we present the results for the invariant operator, the eigenfunctions, and the eigenvalues for the generalized harmonic oscillator[4] whose Hamiltonian is given by:

$$H(x,t) = \beta_1(t)\frac{p^2}{2m} + \beta_2(t)\frac{\omega_0}{2}[x,p]_+ + \beta_3(t)\frac{m}{2}\omega_0^2 x^2 \tag{7}$$

where $\beta_1(t)$, $\beta_2(t)$, and $\beta_3(t)$ are real functions of time. Applying the invariance law (1) to the Hamiltonian (7), one obtains for $I(x, t)$ the following expression:

$$I(x,t) = \frac{1}{\beta_1}\left\{ \beta_1^2\sigma^2 p^2 - m\beta_1\Lambda\sigma^2[x,p]_+ + \left(\frac{1}{\sigma^2} + m^2\Lambda^2\sigma^2\right)x^2 \right\}$$

or

$$I(x,t) = \left[\beta_1\sigma^2\left(p - \frac{m\Lambda}{\beta_1}x\right)^2 + \frac{x^2}{\beta_1\sigma^2} \right] \tag{8}$$

where the function $\Lambda(t)$ is given by

$$\Lambda(t) = \frac{\dot{\sigma}}{\sigma} + \frac{\dot{\beta}_1}{2\beta_1} - \omega_0\beta_2 \tag{9}$$

The function $\sigma(t)$ is a *real function* which is a solution of Pinney's differential equation

$$\ddot{\sigma} + \Omega^2(t)\sigma = \frac{1}{m^2\sigma^3} \tag{10}$$

where $\Omega^2(t)$ is given by the expression

$$\Omega^2(t) = \omega_0^2(\beta_1\beta_3 - \beta_2^2) + \omega_0 \frac{\beta_2\dot{\beta}_1 - \dot{\beta}_2\beta_1}{\beta_1} + \frac{\ddot{\beta}_1}{2\beta_1} - \frac{3}{4}\frac{\dot{\beta}_1^2}{\beta_1^2} \tag{11}$$

and the initial conditions of (10) are given by

$$\sigma(0) = (m\omega_0)^{-1/2} \tag{12a}$$

$$\left\{\frac{d}{dt}\sigma(t)\right\}\bigg|_{t=0} = -\frac{1}{2}(m\omega_0)^{-1/2}\left\{\frac{d}{dt}\beta_1(t)\right\}\bigg|_{t=0} \tag{12b}$$

3. THE HARMONIC OSCILLATOR

Let us start with the example of the time-dependent harmonic oscillator given by the hamiltonian[1,4]

$$H(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\Omega^2(t)\hat{x}^2 \tag{13}$$

Indeed the time-independent solution of the harmonic oscillator is found in all textbooks of quantum mechanics. If $\Omega(0) = \omega_0$, then equation (13) becomes

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{x}^2 \tag{14}$$

and the spectrum and the stationary wave function $H_0\psi_n^0 = \epsilon_n^0\psi_n^0$ are well known:

$$\epsilon_n^0 = \hbar\omega_0\left(n + \frac{1}{2}\right) \tag{15}$$

$$\psi_n^0(x, t) = \exp\left[-i\omega_0\left(n + \frac{1}{2}\right)t\right](\pi a_0^2)^{-1/4} \times (2^n n!)^{-1/2}\exp\left(-\frac{x^2}{2a_0^2}\right)\mathcal{H}_n\left(\frac{x}{a_0}\right) \tag{16}$$

where $a_0^2 = \hbar/m\omega_0$ and $\mathcal{H}_n(\xi)$ are the Hermite polynomials. Now let us look for the solution of the time-dependent Schrödinger equation $H(t)\psi_n = i\hbar\partial_t\psi_n$:

$$\psi_n(x, \sigma(t)) = \exp\left[-i\left(n + \frac{1}{2}\right)\omega_0 \int^t \frac{ds}{m\omega_0\sigma^2(s)}\right]$$
$$[\pi a_0^2 m\omega_0\sigma^2(t)]^{-1/4}(2^n n!)^{-1/2} \exp\left[-\frac{x^2}{2a_0^2}\frac{1 - im\sigma(t)\dot{\sigma}(t)}{m\omega_0\sigma^2(t)}\right]$$
$$\mathcal{H}_n\left(\frac{x}{a_0}[m\omega_0\sigma^2(t)]^{-1/2}\right) \tag{17}$$

provided that σ satisfies the classical equation of motion

$$m\ddot{\sigma} + m\Omega^2(t)\sigma = \frac{1}{m\sigma^3} \tag{18}$$

This coupled set of equations shows the intimate relationship between the quantum wave function and the classical motion described by the latter ordinary nonlinear (but linearizable) differential equation.

One can also write the function ψ_n given by (17) using a change of variables which makes more apparent its relationship with the squeezing phenomenon.[4,5] Consider now the same function given by equation (17), but now written as

$$\psi_n(x, \xi(t)) = \exp\left[-i\left(n + \frac{1}{2}\right)\omega_0 \int^t ReC(t)\right]$$
$$\times (2^n n!)^{-1/2}\left(\frac{\pi a_0^2}{ReC(t)}\right)^{-1/4}$$
$$\times \exp\left[-C(t)\frac{x^2}{2a_0^2}\right]\mathcal{H}_n\left([ReC(t)]^{1/2}\frac{x}{a_0}\right) \tag{19}$$

Here $C(t)$ is given by

$$C(t) = \frac{1 - \xi(t)}{1 + \xi(t)} \tag{20}$$

and ξ satisfies the following classical equation of motion:

$$\dot{\xi} + \frac{i\omega_0}{2}\left(\frac{\Omega^2(t)}{\omega_0^2} - 1\right)(1 + \xi^2) + \left(\frac{\Omega^2(t)}{\omega_0^2} + 1\right)\xi = 0 \tag{21}$$

However, in this second version the squeezing phenomenon appears more evident, as $\xi(t)$ is directly related to the squeezing parameter. There exists a quite obvious relationship between $\xi(t)$ and $\sigma(t)$,

$$\xi(t) = \frac{(m\omega\sigma^2(t) - 1) + im\sigma(t)\dot{\sigma}(t)}{(m\omega\sigma^2(t) + 1) - im\sigma(t)\dot{\sigma}(t)} \tag{22}$$

or conversely

$$\frac{1 - |\xi(t)|^2}{[1 + \xi(t)][1 + \xi^*(t)]} = [m\omega_0\sigma^2(t)]^{-1} \tag{23}$$

To see the relationship to the squeezed vacuum, consider the operator defined as

$$S(\beta)|0\rangle = \exp\left(\frac{\beta}{2} a^{+2} - \frac{\beta^*}{2} a^2\right)|0\rangle \tag{24}$$

where $|0\rangle$ is the ground state of the time-independent harmonic oscillator with frequency defined as $\Omega(t = 0) = \omega_0$.

If $\beta(t)$ is written as

$$\beta(t) = r(t) \exp[i\varphi(t)] \tag{25}$$

we now define

$$\xi(t) = \tanh r(t) \exp[i\varphi(t)] \tag{26}$$

Then $S(\beta)$ can be written as

$$S(\beta) = \exp\left(\frac{\xi(t)}{2} a^{+2}\right) \exp\left[\frac{\gamma(t)}{2}\left(a^+a + \frac{1}{2}\right)\right] \exp\left(-\frac{\xi^*(t)}{2} a^2\right) \tag{27}$$

where $\gamma(t) = \ln(1 - |\xi(t)|^2)$ and the operator $S(\beta)$ can be applied not only to the $|0\rangle$-ground state but to any excited state $|n\rangle$ in the form:

$$S(\beta) \exp\left[\frac{i}{2} h(t)\left(a^+a + \frac{1}{2}\right)\right]|n\rangle \tag{28}$$

where $h(t)$ is a function of just $\xi(t)$ and $\Omega(t)$.

4. THE CONFORMAL OSCILLATOR

Another example closely related to the previous one is the wave function solution of $H\psi_n = i\hbar\partial_t\psi_n$, where now the time-dependent Hamiltonian H is given by[10]

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\Omega^2(t)\hat{x}^2 + \frac{\hbar^2 g^2}{2m\hat{x}^2} \tag{29}$$

Here g is a dimensionless coupling constant. The solution is

$$\begin{aligned}\psi_n(x, \sigma(t)) = \exp&\left[-2i(n + r_0)\omega_0 \int^t \frac{ds}{m\omega_0\sigma^2(s)} \right] \\ &\times [a_0^2 m\omega_0\sigma^2(t)]^{-1/4} \left(\frac{2\Gamma(n+1)}{\Gamma(n + 2r_0)} \right)^{1/2} \\ &\times \left\{ \frac{x}{a_0} [m\omega_0\sigma^2(t)]^{-1/2} \right\}^{2r_0 - 1/2} \\ &\times \exp\left[-\frac{x^2}{2a_0^2} \frac{1 - im\sigma(t)\dot{\sigma}(t)}{m\omega_0\sigma^2(t)} \right] \\ &\times \mathcal{L}_n^{2r_0-1}\left(\frac{xs2}{a_0^2 m\omega_0\sigma^2(t)} \right)\end{aligned} \tag{30}$$

The solution (30) holds if and only if $\sigma(t)$ satisfies the classical equation of motion (18):

$$m\ddot{\sigma} + m\Omega^2(t)\sigma = \frac{1}{m\sigma^3} \tag{31}$$

and $\mathcal{L}_n^k(\xi)$ are the associated Laguerre polynomials which satisfy the differential equation

$$\xi\mathcal{L}_n^{\prime\prime k}(\xi) + (k + 1 - \xi)\mathcal{L}_n^{\prime k}(\xi) + n\mathcal{L}_n^k(\xi) = 0 \tag{32}$$

Recall that k (or $2r_0 - 1$) is not necessarily an integer in (32), as may be the case in the present quantum system. The real number r_0 is defined as

$$r_0 = \frac{1}{2}\left[1 + \left(g^2 + \frac{1}{4} \right)^{1/2} \right] \tag{33}$$

One could also calculate the squeezing factors by rewriting the wave function as been done in the previous section for the time-dependent harmonic oscillator.

5. THE INVARIANT OPERATOR AND THE CANONICAL TRANSFORMATIONS

Case I. The Time-Dependent Harmonic Oscillator

One of the main goals of this paper is to show how to generate the same invariant operators using quantum canonical transformations.[6,7] This development leads to a new way to analyze the same problems of time-dependent Hamiltonians and extend them to a much larger class encompassing new and interesting physical systems. We shall use particular cases of the generalized harmonic oscillator to illustrate how this new method works. Let us first take in equation (7) the following particular choice of initial conditions: $\{\beta_1(0) = 1, \beta_2(0) = 0, \beta_3(0) = 1\}$. For any time $t > 0$ these functions take the values $\{\beta_1(t) = 1, \beta_2(t) = 0, \beta_3(t) = \beta_3(t)\}$. The Hamiltonian $H(x, 0)$ takes the form

$$H_1(x) = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x^2 \tag{34}$$

One can systematically construct the following operator:

$$W_1(x, t) = \exp\left\{ -\frac{i}{4\hbar} \log(m\omega_0\sigma^2)[x, p]_+ \right\} \exp\left(i \frac{m^2\omega_0\dot{\sigma}\sigma}{2\hbar} x^2 \right) \tag{35}$$

The details of the rigorous mathematical construction will be given elsewhere. We shall try to avoid technical details in this paper in order to emphasize the physical usefulness of the procedure. Using $W_1(x, t)$ one can obtain $I(x, t)$ in a much simpler form:

$$\begin{aligned} I(x, t) &= W_1(x, t)\left(\frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x^2 \right) W_1^{+}(x, t) \\ &= \beta_1\sigma^2 \left(p - \frac{m\Lambda}{\beta_1} x \right)^2 + \frac{x^2}{\beta_1\sigma^2} \end{aligned} \tag{36}$$

which obviously coincides with the expression as promised. The spectral problem for $I(x, t)$ can easily be solved in the form

$$I(x, t)\Phi_n(x, t) = \epsilon_n \Phi_n(x, t)$$

where $\epsilon_n = \hbar(n + \frac{1}{2})$ and the eigenfunctions $\Phi_n(x, t)$ are explicitly given by the expression (17):

$$\Phi_n(x, t) = [2^{2n}(n!)^2\pi\hbar\sigma^2]^{-1/4} \exp\left[(im\sigma\dot{\sigma} - 1) \frac{\xi^2}{2} \right] \mathcal{H}_n(\xi)$$

where

$$a_0^2 = \frac{\hbar}{m\omega_0} \qquad \text{and} \qquad \xi = \frac{x}{a_0} (m\omega_0\sigma^2)^{-1/2}$$

and $\mathcal{H}_n(\xi)$ are the Hermite polynomials of variable ξ. The *Lewis phases* can now be readily calculated with the help of equation (6). After some tedious calculation we find for $\alpha_n(t)$ the *exact* expression[3]

$$\alpha_n(t) = -\left(n + \frac{1}{2}\right)\int_0^t \frac{ds}{m\sigma^2(s)} \tag{37}$$

The last step is to find the set of orthonormal eigenfunctions solving the spectral problem for the time dependent Schrödinger equation with Hamiltonian $H(x, t)$ given by (7) with $\{\beta_1(t) = 1, \beta_2(t) = 0, \beta_3(t) = \beta_3(t)\}$. As we know from the discussion of the previous section and equation (5), these eigenfunctions are constructed as

$$\Psi_n(x, t) = \exp\left[-i\left(n + \frac{1}{2}\right)\int_0^t \frac{ds}{m\sigma^2(s)}\right]\Phi_n(x, t) \tag{38}$$

where the function $\sigma(t)$ satisfies Pinney's equation:

$$m\ddot{\sigma} + m\Omega^2(t)\sigma = \frac{1}{m\sigma^3} \tag{39}$$

with $\Omega^2(t) = \omega_0^2\beta_3(t)$.

6. THE INVARIANT OPERATOR AND THE CANONICAL TRANSFORMATIONS

Case II. The Infinite Square Well With One Moving Boundary

Let us now take in equation (7) the following choice of initial conditions: $\{\beta_1(0) = 1, \beta_2(0) = \beta_3(0) = 0\}$. The Hamiltonian $H(x, 0)$: now takes the form

$$H_2(x) = \frac{p^2}{2m} \tag{40}$$

Next we define $W_2(x, t)$ like $W_1(x, t)$, but this time identifying

$$L(t) = (m\omega_0)^{1/2}L_0\sigma(t) \tag{41}$$

Notice that for the initial conditions for $\sigma(t)$ given by expressions (12a) and (12b) we obtain a physical set of initial conditions for $L(t)$, namely $L(0) = L_0$ and $\dot{L}(0) = 0$. The operator $W_2(x, t)$ now reads

$$W_2(x, t) = \exp\left\{-\frac{i}{2\hbar}\log\left(\frac{L(t)}{L_0}\right)[x, p]_+\right\}\exp\left(\frac{imL(t)\dot{L}(t)}{2\hbar L_0^2}x^2\right) \tag{42}$$

Next we set $W_2(x, t)H_2(x)W_2^\dagger(x, t) = I_2(x, t)$ and in so doing we get the following invariant for this case:

$$I_2(x, t) = W_2(x, t) \frac{p^2}{2m} W_2^{+}(x, t) = \frac{L^2(t)}{2mL_0^2} \left(p - m \frac{\dot{L}(t)}{L(t)} x \right)^2 \tag{43}$$

with eigenfunctions and eigenvalues given respectively by

$$\Phi_n(x, t) = \sqrt{\frac{2}{L(t)}} \exp\left(\frac{im\dot{L}(t)}{2\hbar L(t)} x^2 \right) \sin\left(\frac{n\pi x}{L(t)} \right) \tag{44}$$

and

$$\epsilon_n = \frac{\hbar^2 \pi^2 n^2}{2mL_0^2} \tag{45}$$

Note, however, that we still do not know whether this invariant is associated with a given time-dependent Hamiltonian $H^*(x, t)$. Nevertheless our formalism yields unambiguously this Hamiltonian under the obvious assumption that the operator $I_2(x, t)$ is its associated invariant. Formally this is equivalent to saying that both operators are linked through the following equation:

$$\frac{dI_2(x, t)}{dt} = \frac{\partial I_2(x, t)}{\partial t} + \frac{1}{i\hbar} [I_2(x, t), H^*(x, t)] \tag{46}$$

where $I_2(x, t)$ is now given by (43):

$$I_2(x, t) = \frac{L^2(t)}{2mL_0^2} \left(p - m \frac{\dot{L}(t)}{L(t)} x \right)^2 \tag{47}$$

A simple calculation shows that $H^*(x, t)$ takes the following form:

$$H^*(x, t) = \frac{p^2}{2m} - \frac{1}{2} m \frac{\ddot{L}}{L} x^2 \tag{48}$$

The *Lewis phase* α_n^* can be easily calculated with expression (6) and $\Phi_n(x, t)$ and $H^*(x, t)$ given respectively by (44) and (48). The result is

$$\alpha_n^*(t) = -\frac{n^2\pi^2\hbar}{2m} \int_0^t \frac{ds}{L^2(s)} \tag{49}$$

and the wave functions for the hamiltonian (48) take the final form

$$\Psi_n(x, t) = \sqrt{\frac{2}{L(t)}} \exp\left(-\frac{in^2\pi^2\hbar}{2m} \int_0^t \frac{ds}{L^2(s)} \right) \times \exp\left(\frac{im\dot{L}(t)}{2\hbar L(t)} x^2 \right) \sin\left(\frac{n\pi x}{L(t)} \right) \tag{50}$$

The conclusion is that the boundary generates an "effective interaction" due to the time-dependent boundary conditions. This interaction can only be seen when the canonical formalism is systematically applied. Therefore even if one begins with the free-particle Hamiltonian the system presents an interaction potential just due to the time-dependent character of the boundary conditions. The implications of this conclusion for a wide variety of physical situations need not be emphasized (see also Refs. 8 and 9).

7. CANONICAL TRANSFORMATIONS IN THE HAMILTONIAN AND GAUGE TRANSFORMATIONS

Case I. The Time-Dependent Harmonic Oscillator

Let us now consider the new wave function

$$\Psi_n(x, t) = G(x, t)\Psi_n(x, t) \tag{51}$$

The new Hamiltonian $\hat{H}^*(t)$ such that

$$i\hbar\partial_t\Psi_n(x, t) = \hat{H}^*(t)\Psi_n(x, t) \tag{52}$$

is obviously related to $H^*(t)$ in the well-known form

$$\hat{H}^*(t) = G(x, t)H^*(x, t)G^+(x, t) - i\hbar G(x, t)\dot{G}^+(x, t) \tag{53}$$

The gauge transformation for the canonical operators p and x in the Hamiltonian $\hat{H}^*(t)$ takes the form

$$p \rightarrow p - \frac{\partial Q(x, t)}{\partial x} \tag{54a}$$

$$V(x, t) \rightarrow V(x, t) - \frac{\partial Q(x, t)}{\partial t} \tag{54b}$$

The use of the canonical formalism in time-dependent quantum systems not only leads to an efficient way to find the exact time evolution operators, but is also linked to the idea of minimal coupling lying behind the apparent gauge principle we have just described. Let us now apply these ideas to the time-dependent harmonic oscillator whose wave function is given by (17):

$$\Psi_n(x, \sigma(t)) = \exp\left[-i\left(n + \frac{1}{2}\right)\omega_0 \int^t \frac{ds}{m\omega_0\sigma^2(s)}\right]$$
$$\times [\pi a_0^2 m\omega_0\sigma^2(t)]^{-1/4}(2^n n!)^{-1/2} \exp\left[-\frac{x^2}{2a_0^2}\frac{1 - im\sigma(t)\dot{\sigma}(t)}{m\omega_0\sigma^2(t)}\right]$$

$$\times \mathcal{H}_n\left(\frac{x}{a_0}[m\omega_0\sigma^2(t)]^{-1/2}\right) \tag{55}$$

Let us now define $\Psi(x, t)$ according to (51) as

$$\begin{aligned}\Psi_n(x, t) &= G(x, t)\Psi_n(x, t)\\ &= \exp\left[\frac{i}{\hbar}Q(x, t)\right] = \exp\left[-\frac{i}{\hbar}\frac{m\dot{\sigma}(t)}{2\sigma(t)}x^2\right]\end{aligned} \tag{56}$$

Applying (53) to the Hamiltonian (13) and using also the constraint (18), we obtain

$$\begin{aligned}\hat{H}^*(t) = &\frac{1}{2m}\left[p + m\frac{\dot{\sigma}(t)}{\sigma(t)}x\right]^2\\ &-\frac{m}{2}\left[\frac{\dot{\sigma}(t)}{\sigma(t)} + \frac{1}{m\sigma^2}\right]\left[\frac{\dot{\sigma}(t)}{\sigma(t)} - \frac{1}{m\sigma^2}\right]x^2\end{aligned} \tag{57}$$

This transformed Hamiltonian can also be obtained with the help of the gauge transformation (54a), (54b). Its wave functions are given by

$$\begin{aligned}\Psi_n(x, \sigma(t)) = &\exp\left[-i\left(n + \frac{1}{2}\right)\omega_0\int^t \frac{ds}{m\omega_0\sigma^2(s)}\right]\\ &\times [\pi a_0^2 m\omega_0\sigma^2(t)]^{-1/4}(2^n n!)^{-1/2}\exp\left\{-\frac{x^2}{2a_0^2}[m\omega_0\sigma^2(t)]^{-1}\right\}\\ &\times \mathcal{H}_n\left(\frac{x}{a_0}[m\omega_0\sigma^2(t)]^{-1/2}\right)\end{aligned} \tag{58}$$

The wave functions (58) look exactly like those of the stationary harmonic oscillator, but rescaling the time-independent length in the form $a_0^2 \to a_0^2 m\omega_0\sigma^2(t)$, where now $\sigma(t)$ satisfies

$$m\ddot{\sigma} + m\Omega^2(t)\sigma = \frac{1}{m\sigma^3} \tag{59}$$

8. CANONICAL TRANSFORMATIONS IN THE HAMILTONIAN AND GAUGE TRANSFORMATIONS

Case II. The Infinite Square Well with One Moving Boundary

Let us now consider the Hamiltonian and wave function of the infinite square well with a moving boundary described in a previous section and given respectively by the expressions

$$H^*(x, t) = \frac{p^2}{2m} - \frac{1}{2} m \frac{\ddot{L}}{L} x^2 \tag{60}$$

$$\Psi_n(x, t) = \sqrt{\frac{2}{L(t)}} \exp\left(-\frac{in^2\pi^2\hbar}{2m} \int_0^t \frac{ds}{L^2(s)} \right) \times \exp\left(\frac{im\dot{L}(t)}{2\hbar L(t)} x^2 \right) \sin\left(\frac{n\pi x}{L(t)} \right) \tag{61}$$

We now look for a new wave function $\hat{\Psi}_n(x, t)$ related to $\Psi_n(x, t)$ through the relationship $\hat{\Psi}_n(x, t) = G(x, t)\Psi_n(x, t)$, where $G(x, t)$ is given by

$$G(x, t) = \exp\left(-\frac{im}{2\hbar} \frac{\dot{L}(t)}{L(t)} x^2 \right) \tag{62}$$

Indeed, this canonical transformation is a quantum gauge transformation of the form

$$G(x, t) = \exp\left[\frac{i}{\hbar} Q(x, t) \right] \tag{63}$$

whose unitary generator is given by

$$Q(x, t) = -\frac{1}{2} m \frac{\dot{L}(t)}{L(t)} x^2 \tag{64}$$

Applying now

$$\hat{H}^*(t) = G(x, t)H^*(x, t)G^+(x, t) - i\hbar G(x, t)\dot{G}^+(x, t) \tag{65}$$

one can obtain the new Hamiltonian $\hat{H}^*(t)$ starting with the expressions for $H^*(t)$ and $G(x, t)$ given by (60) and (62). The final result of this calculation yields the following expression for $\hat{H}^*(t)$:

$$\hat{H}^*(t) = \frac{1}{2m}\left(p + m \frac{\dot{L}(t)}{L(t)} x \right)^2 - \frac{1}{2} m \frac{\dot{L}^2(t)}{L^2(t)} x^2 \tag{66}$$

with wave functions $\hat{\Psi}_n(x, t)$ of the form

$$\hat{\Psi}_n(x, t) = \sqrt{\frac{2}{L(t)}} \exp\left(-\frac{in^2\pi^2\hbar}{2m} \int_0^t \frac{ds}{L^2(s)} \right) \sin\left(\frac{n\pi x}{L(t)} \right) \tag{67}$$

The physical interpretation of these equivalent quantum systems described alternatively by the set of Hamiltonian and wave function (60)–(61) or (66)–(67), representing the quantum behavior of a particle confined in a one-dimensional impenetrable box, requires further discussion. The time dependence comes from the motion of the boundary. However, this motion cannot be described as a force included in the Hamiltonian as it is due to some external action on the wall from outside the box. There the wave function of the particle vanishes everywhere and no physical operator representing a force acting on the exterior of the well can be included in the Hamiltonian in a rigorous way, as the Hamiltonian deals with the physical description of the particle inside the box. If we call $\Omega(t)$ the external frequency acting on the wall from *outside*, its action *must cancel* the motion of $L(t)$ as seen from *inside*, in the form

$$\ddot{L}(t) + \Omega^2(t)L(t) = 0 \tag{68}$$

which also represent a sort of classical equation of motion for the motion of the wall as in the previous examples.

Turning our attention to the problem of squeezing, it is worth noting that the invariant operator can be found by applying a squeezing operator of the form $S(\beta)$ to the free-particle Hamiltonian. One can also find the exact fluctuation spectrum and the degree of squeezing. The nontrivial observation is, however, the form of the wave function arising from a classical time-dependent constraint. After applying the squeezing operator, one sees that instead of the purely free particle Hamiltonian, one is actually solving the *effective Hamiltonian* given by the expression

$$H = \frac{\hat{p}^2}{2m} - \frac{1}{2} m \frac{\ddot{L}(t)}{L(t)} \hat{x}^2 \tag{69}$$

The fluctuations in the measurement of position and momentum can also be used to give an idea on the squeezing factor, which would now be coming just from the motion of the walls, as already discussed:

$$(\Delta x)^2 = \frac{L^2(t)}{12}\left(1 - \frac{6}{n^2\pi^2}\right) \tag{70a}$$

$$(\Delta p)^2 = \frac{\hbar^2\pi^2 n^2}{L^2(t)} + \frac{m^2\dot{L}^2(t)}{12}\left(1 - \frac{6}{n^2\pi^2}\right) \tag{70b}$$

9. CONCLUSIONS

A systematic formalism for dealing with the dynamics of nonrelativistic time-dependent quantum Hamiltonians has been presented. We have checked the formalism using various examples. In the case of the generalized time-dependent harmonic oscillator, we found all information including quantum phases by systematically applying the formalism. New information appears in the case of the time-dependent potential well of infinite height.

The boundary in motion generates a nontrivial interaction which manifests itself as an effective potential which depends in turn on the classical motion of the wall. Applications to physical systems confined in nonstationary boxes or bags, conformal oscillators with external random forces,[10,12] Fermi oscillators, various aspects of chaos in quantum physics,[11–13] and recent proposals for the rigorous description of Bose–Einstein condensation[14] are areas which could benefit from the results discussed. A preliminary announcement of the results discussed in this paper has appeared.[15] A similar approach with different emphasis on the construction of the canonical invariant operators has also recently been discussed by Lejarreta.[16]

A final point should be made with regard to the realistic nonlocal description of the physical world (for an interesting account of this subject see the review in ref. 2). If the classical constraints were of purely nonlinear nature, the wave function would still be a solution of the linear Schrödinger equation, but with a "pilot" following these nonlinear classical rules. The "nonlinearity" of quantum mechanics would appear in an interesting way as a realistic nonlocal description of nature.

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